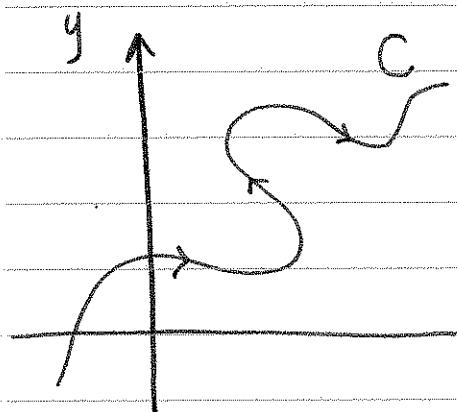


Chapter 10: Parametric Equations & Polar Coordinates

Section 10.1: Curves defined By Parametric Equations



Imagine a particle moving along a Curve C as in the Figure on the left. It would be impossible to represent the Curve C as $y = f(x)$ (y as a function of x), because the Curve fails the Vertical line test. But the x and y-coordinates of the

Particle are functions of time, t. So we can write

$x = f(t)$, and $y = g(t)$; We call t the "parameter".
for every value of t, we get a value for x and y:

$(x, y) = (f(t), g(t))$, which represents a Unique point on the x-y plane.

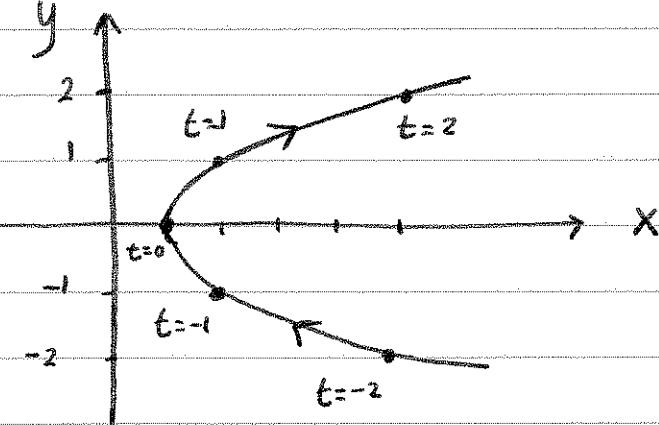
Note: t doesn't necessarily Represent time; But in the Context of a Particle, it often is time.

The equations $x = f(t)$, $y = g(t)$ are called "parametric equations"; The curve C is called a "parametric curve".

Example ① Sketch and identify the curve defined by the parametric Equations $x = t^2 + 1$, $y = t$.

Let's calculate a few values for x and y , for different values of t , and plot them:

t	x	y
-2	5	-2
-1	2	-1
0	1	0
1	2	1
2	5	2



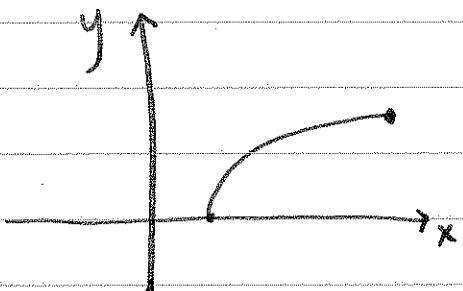
it looks like the curve traced out by the particle is a Parabola. Indeed, if $x = t^2 + 1$, and $y = t$, Then we have $x = y^2 + 1$, which is the equation of a parabola in the horizontal direction.

. Note That here, we did not place a restriction on "t".

Sometimes, we restrict t to lie in a finite interval;

for instance we could write $x=t^2+1$, $y=t$, $0 \leq t \leq 2$.

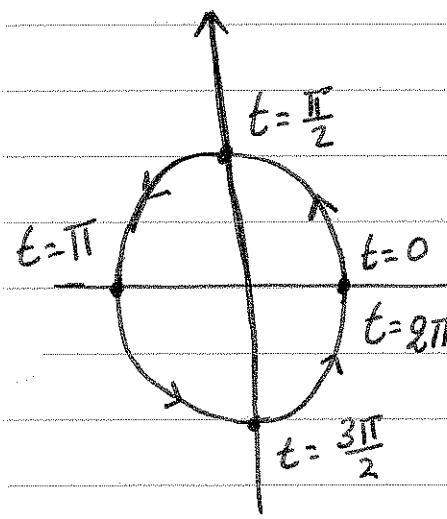
in this case, we only get a finite piece of the curve C:



Example ②: What curve is represented by the parametric equations $x = \cos t$, $y = \sin t$, $0 \leq t \leq 2\pi$?

Observe that $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$. Thus the point (x, y) moves on the unit circle $x^2 + y^2 = 1$, as t

increases. We can interpret t as the angle; if



t changes from 0 to 2π , the particle moves along the unit circle only once! (In the counter-clockwise direction).

Example③ What Curve is Represented by the parametric

Equations $x = \sin 2t$, $y = \cos 2t$, $0 \leq t \leq 2\pi$?

We still have $x^2 + y^2 = \sin^2 2t + \cos^2 2t = 1$, so, a Unit Circle; But 2 things are different:

① the angle is $2t$; for $0 \leq t \leq 2\pi$, we have

$0 \leq 2t \leq 4\pi$; So we move along the Unit Circle

TWICE!

② We now start at the point $(\sin 0, \cos 0) = (0, 1)$, and

move along the Circle in a Clockwise direction!

Example④: How do we represent a circle centered at

(h, k) with radius r , using parametric Equations?

. We want to find $x = f(t)$, $y = g(t)$, where $0 \leq t \leq 2\pi$.

Suppose for a moment the center is $(0, 0)$. Then

$x^2 + y^2 = r^2$: So, let $x = r \cos t$, $y = r \sin t$.

Then $r^2 \cos^2 t + r^2 \sin^2 t = r^2$, as needed.

if the center is (h, k) , we simply translate the circle at the origin h units in the x -direction and k units in the y -direction. So, we get

$$x = h + r \cos t, \quad y = k + r \sin t, \quad 0 \leq t \leq 2\pi.$$

Example 5: Find a parametrization of the line segment starting at the point (x_0, y_0) and ending at the point (x_1, y_1) ,



where $0 \leq t \leq 1$; Use affine functions

$x(t)$ and $y(t)$:

(f is affine if $f(t) = at + b$, a, b constants)

We want $x = f(t) = at + b$, where $f(0) = x_0$, $f(1) = x_1$;

$$\text{So, } x = f(t) = x_0 + t(x_1 - x_0).$$

We want $y = g(t) = ct + d$, where $g(0) = y_0$, $g(1) = y_1$.

$$\text{So, } y = g(t) = y_0 + t(y_1 - y_0). \quad \text{Therefore,}$$

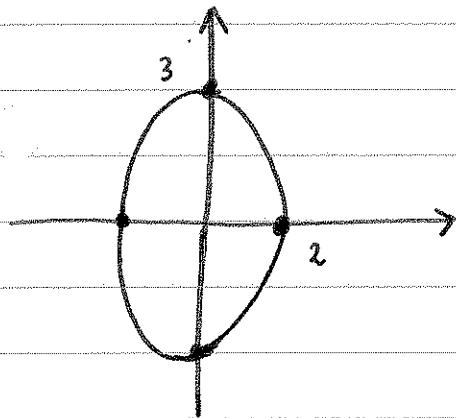
$$(x, y) = (x_0 + t(x_1 - x_0), y_0 + t(y_1 - y_0)), \quad 0 \leq t \leq 1.$$

Example ⑥: Find a parametrization of the bottom part of

the curve $\frac{x^2}{4} + \frac{y^2}{9} = 1$, which starts at $(x,y) = (2,0)$ and ends at $(x,y) = (-2,0)$; here $0 \leq t \leq \pi$.

Observe that $\frac{x^2}{4} + \frac{y^2}{9} = 1$ is the equation of an ellipse

centered at $(0,0)$, with a horizontal radius $a = \sqrt{4} = 2$, and vertical radius $b = \sqrt{9} = 3$. Here's a plot:



Think of an ellipse as a generalized circle. We still use trig. functions for x and y . We have two choices:

$$\begin{aligned} \text{① } & x = 2\sin t, \quad y = 3\cos t \\ & \left. \right\} \text{ in both cases, } \frac{x^2}{4} + \frac{y^2}{9} = 1 \end{aligned}$$

$$\begin{aligned} \text{② } & x = 2\cos t, \quad y = 3\sin t. \end{aligned} \left. \right\} \text{ By the pythagorean identity.}$$

In Case ① we start at $(2\sin(0), 3\cos(0)) = (0,3)$ Incorrect.

In Case ② we start at $(2\cos(0), 3\sin(0)) = (2,0)$ and end at

$(2\cos(\pi), 3\sin(\pi)) = (-2,0)$. Also, at $t = \frac{\pi}{2}$, we are at the

Point $(2\cos\frac{\pi}{2}, 3\sin\frac{\pi}{2}) = (0,3)$. This is a problem.

The point $(0,3)$ is not in the bottom part of the ellipse.

We can fix this easily. Since at $\frac{\pi}{2}$ we want to be at $(0,-3)$

let $x = 2 \cos t$, $y = -3 \sin t$, $0 \leq t < \pi$. Then we
are done.